ON THE CONSTRUCTION OF F-SQUARES AND SINGLE DEGREE-OF-FREEDOM CONTRASTS

(Preliminary)

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Abstract

The orthogonality present among contrasts in the analysis of variance of a two-factor factorial has not been utilized in the construction of F-squares and latin squares for squares of order n not necessarily a prime power. An attempt is made in this direction herein. Weaknesses of linear models theory in this area is illustrated and several unsolved problems are posed. The procedure is detailed for n = 3, 4, and 6. One of the new results is the construction of seven orthogonal $F(6;A_1^2,A_2^2,A_3^2)$-squares and one latin square, or $F(6;A_1,A_2,A_3,A_4,A_5,A_6)$-square. Thus, 7(3-1) + (6-1) = 19 of the 25 row by column interaction sum of squares are obtained. This leaves only six interaction degrees of freedom remaining before a complete set of orthogonal F-squares of order six is obtained.

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1. Introduction

This paper is written to set forth some ideas on the construction of latin squares and F-squares utilizing orthogonality of contrasts in the analysis of variance. These ideas may be implied from Fisher and Yates [1963] and Federer *et al.* [1971]. The work reported here is related to that with D.A. Anderson and E. Seiden [1974] and with current research with these individuals and F.-C. Helen Lee, T. Graves, and J.P. Mandeli. The ideas of the author are set forth in order that the above named individuals may utilize these results in current research.

The use of orthogonal degree-of-freedom contrasts to construct latin squares and F-squares of order n, and vice versa, is expounded in section 2 for n = 3, in section 4 for n = 5, in section 5 for n = a prime power, and in sections 5 and 6 for n = 6. The seventh section contains some unsolved problems in linear models theory which are needed in the construction of orthogonal latin squares and F-squares.

2. The Latin Square of Order 3

Given the following two orthogonal Latin squares of order 3 with the rows being denoted as levels of factor a and the columns being denoted as levels of factor b,

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$LS_1(3)$	levels of b		
levels of a	0	1	2
0	α	β	γ
1	β	γ	α
2	γ	α	β

$\perp$

$LS_2(3)$	levels of b		
levels of a	0	1	2
0	I	II	III
1	III	I	II
2	II	III	I

An analysis of variance (ANOVA) is:

Source of variation	df	
Total	9	
Correction for mean	1	
Rows = A effect	2	
$R_1 = a_2$ vs. a_0		1
$R_2 = a_0 + a_2$ vs. $2a_1$		1
Columns = B effect	2	
$C_1 = b_2$ vs. b_0		1
$C_2 = b_0 + b_2$ vs. $2b_1$		1
Treatments from $LS_1(3)$ = AB	2	
$C_2 = AB_2$ vs. $AB_0 = \gamma$ vs. α		1
$C_3 = AB_2 + AB_0$ vs. $2AB_1$ $= \alpha + \gamma$ vs. 2β		1
Treatments from $LS_2(3)$	2	
$C_4 = AB_1^2$ vs. $AB_0^2 =$ III vs. I		1
$C_5 = AB_1^2 + AB_0^2$ vs. AB_2^2 = I + III vs. 2II		1

Thus the contrasts C_2 and C_3 and C_4 and C_5 and their interactions produce the eight degrees of freedom as follows:

Contrast		00	01	02	10	11	12	20	21	22
		Row and Column Level								
Mean = C_1		+	+	+	+	+	+	+	+	+
$LS_1(3)$	C_2	-	0	+	0	+	-	+	-	0
	C_3	+	-2	+	-2	+	+	+	+	-2
$LS_2(3)$	C_4	-	0	+	+	-	0	0	+	-
	C_5	+	-2	+	+	+	-2	-2	+	+
Row and Column	$C_2 \times C_4$	+	0	+	0	-	0	0	-	0
	$C_2 \times C_5$	-	0	+	0	+	2	-2	-	0
	$C_3 \times C_4$	-	0	+	-2	-	0	0	+	2
	$C_3 \times C_5$	+	4	+	-2	+	-2	-2	+	-2

If the yield equation without the error term ϵ_{hij} is $Y_{hij} = \mu + \rho_h + \gamma_i + \tau_j$, then contrasts,

$$C_2 \times C_4 = Y_{00\alpha I} + Y_{02\gamma III} - Y_{11\gamma I} - Y_{21\alpha III} = 2\rho_0 - \rho_1 - \rho_2 + \gamma_0 + \gamma_2 - 2\gamma_1,$$

$$C_2 \times C_5 = 3(\rho_1 - \rho_2 - \gamma_0 + \gamma_2),$$

$$C_3 \times C_4 = 3(-\rho_1 + \rho_2 - \gamma_0 + \gamma_2), \quad \text{and}$$

$$C_3 \times C_5 = 3(2\rho_0 - \rho_1 - \rho_2 + 2\gamma_1 - \gamma_0 - \gamma_2) \quad .$$

Thus,

$$\begin{bmatrix} 1/2 & 1/6 & 0 & 0 \\ -1/2 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 \\ 0 & 0 & 1/6 & -1/6 \end{bmatrix} \begin{bmatrix} C_2 \times C_4 \\ C_3 \times C_5 \\ C_2 \times C_5 \\ C_3 \times C_4 \end{bmatrix} = \begin{bmatrix} 2\rho_0 - \rho_1 - \rho_2 \\ 2\gamma_1 - \gamma_0 - \gamma_2 \\ \gamma_2 - \gamma_0 \\ \rho_1 - \rho_2 \end{bmatrix}$$

which indicates that a linear combination of the interaction single degree-of-freedom contrasts, produces the row and column contrasts.

Note also that one could reverse the process and take particular row and column contrasts; then, the interactions of the individual row and column contrasts could be obtained; and finally a linear combination of the contrasts would yield the contrasts among treatments in the individual latin squares (or F-squares) in the set of mutually orthogonal latin squares (or F-squares).

For the above example, consider the following row and column contrasts:

Contrast		Row and Column Designation 00	01	02	10	11	12	20	21	22
Mean		+	+	+	+	+	+	+	+	+
Row contrasts:	A_1	-	-	-	+	+	+	0	0	0
	A_2	+	+	+	+	+	+	-2	-2	-2
Column contrasts:	B_1	+	-	0	+	-	0	+	-	0
	B_2	+	+	-2	+	+	-2	+	+	-2
Interactions:	A_1B_1	-	+	0	+	-	0	0	0	0
	A_1B_2	-	-	2	+	+	-2	0	0	0
	A_2B_1	+	-	0	+	-	0	-2	2	0
	A_2B_2	+	+	-2	+	+	-2	-2	-2	4

Now we wish to find what linear combination of row and column interaction contrasts produce treatment contrasts in $LS_1(3)$ and $LS_2(3)$. In particular we wish to solve the following set of equations for the a_{ij}:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} A_1B_1 \\ A_1B_2 \\ A_2B_1 \\ A_2B_2 \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} = \begin{bmatrix} - & 0 & + & 0 & + & - & + & - & 0 \\ + & -2 & + & -2 & + & + & + & + & -2 \\ - & 0 & + & + & - & 0 & 0 & + & - \\ + & -2 & + & + & + & -2 & -2 & + & + \end{bmatrix}$$

and the matrix containing A_1B_1, A_1B_2, A_2B_1, and A_2B_2 is the last four rows of the preceding matrix and has dimensions 4×9. The solution is:

$$\frac{1}{4}\begin{bmatrix} 0 & 2 & -2 & 0 \\ -6 & 0 & 0 & -2 \\ 3 & 1 & 1 & -1 \\ -3 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} A_1B_1 \\ A_1B_2 \\ A_2B_1 \\ A_2B_2 \end{bmatrix} = \begin{bmatrix} AB_2 - AB_0 = C_2 \\ AB_2 + AB_0 \text{ vs. } 2AB_1 = C_3 \\ AB_1^2 \text{ vs. } AB_0^2 = C_4 \\ AB_1^2 + AB_0^2 \text{ vs. } 2AB_2^2 = C_5 \end{bmatrix}$$

If instead of using contrasts C_4 and C_5, one had used $C_6 = AB_1^2$ vs. AB_2^2 and $C_7 = AB_1^2 + AB_2^2$ vs. $2AB_0^2$, then,

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & -1 & 0 \\ -3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} A_1B_1 \\ A_1B_2 \\ A_2B_1 \\ A_2B_2 \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ C_6 \\ C_7 \end{bmatrix}$$

which involves simpler linear contrasts than the previous breakdown of treatment degrees of freedom.

3. Latin Squares of Order 4

For n = 4, one could consider various forms of single degree-of-freedom contrasts. For example, a factorial breakdown for four contrasts would be

	0	1	2	3
C_1	+	+	+	+
C_2	-	+	-	+
C_3	-	-	+	+
C_4	+	-	-	+

,

a polynomial regression set of contrasts would be

	0	1	2	3
C_1'	+	+	+	+
C_2'	-3	-1	1	3
C_3'	1	-1	-1	1
C_4'	1	-3	3	-1

,

and a Helmert-polynomial set of contrasts would be

	0	1	2	3
C_1^*	1	1	1	1
C_2^*	1	-1	0	0
C_3^*	1	1	-2	0
C_4^*	1	1	1	-3

.

Many other sets of orthogonal contrasts among the four numbers 0, 1, 2, and 3, are possible. Since the number of possible sets of row and column contrasts becomes large as n increases and since it is desired to pick as simple a linear combination as is possible, the first procedure described in the previous section could provide the row and column contrasts to use. Note that <u>any</u> set of contrasts may be utilized but that the simplest transformation set should be used if at all possible. However, this procedure does depend upon which set of treatment contrasts is selected.

Federer <u>et</u> <u>al</u>. [1971], page 8, indicate that the factorial selection of row and column contrasts produces the full set of orthogonal latin squares of order 4, i.e., OL(4,3). The 16 observations are related to a 2^4 factorial with A, B, and AB forming the row contrasts, C, D, and CD forming the column contrasts, and the following row by column interactions forming the three mutually orthogonal latin squares of order 4:

	Effects in 2^4 factorial
$LS_1(4)$	AC, BD, ABCD
$LS_2(4)$	BC, ACD, ABD
$LS_3(4)$	AD, ABC, BCD

,

where the latin squares are:

$LS_1(4)$

I	II	III	IV
II	I	IV	III
III	IV	I	II
IV	III	II	I

$LS_2(4)$

W	Z	X	Y
X	Y	W	Z
Y	X	Z	W
Z	W	Y	X

$LS_3(4)$

α	γ	δ	β
β	δ	γ	α
γ	α	β	δ
δ	β	α	γ

and the factorial designation for factors a, b, c, and d is:

	Column			
Row	1	2	3	4
1	0000	0001	0010	0011
2	0100	0101	0110	0111
3	1000	1001	1010	1011
4	1100	1101	1110	1111

The factorial and the polynomial regression breakdowns for row and column contrasts and interactions are:

Source of variation		d.f.		Source of variation	
Total		16		Total	
Correction for mean		1		Correction for mean	
Row contrasts		3			
$A = (2R_L + R_c)/5$	1		1	Rows linear	$= R_L = 2A + B$
$B = (R_L - 2R_c)/5$	1		1	Rows quadratic	$= R_Q = AB$
$AB = R_Q$	1		1	Rows cubic	$= R_c = A - 2B$
Column contrasts		3			
$C = (2C_L + C_c)/5$	1		1	Columns linear	$= C_L = 2C + D$
$D = (C_L - 2C_c)/5$	1		1	Columns quadratic	$= C_Q = CD$
$CD = C_Q$	1		1	Columns cubic	$= C_c = C - 2D$
Roman numbers		3			
$AC = (2R_L + R_c)(2C_L + C_c)/25$	1			1	$R_L C_L$
$BD = (R_L - 2R_c)(C_L - 2C_c)/25$	1			1	$R_L C_Q$
$ABCD = R_Q C_Q$	1			1	$R_L C_c$
Greek letters		3			
$ABD = R_Q (C_L - 2C_c)/5$	1			1	$R_Q C_L$
$BC = (R_L - 2R_c)(2C_L + C_c)/25$	1		9	1	$R_Q C_Q$
$ACD = (2R_L + R_c) C_Q/5$	1			1	$R_Q C_c$
Latin letters		3			
$AD = (2R_L + R_c)(C_L - 2C_c)/25$	1			1	$R_c C_L$
$ABC = R_Q (2C_L + C_c)/5$	1			1	$R_c C_Q$
$BCD = (R_L - 2R_c) C_Q/5$	1			1	$R_c C_c$

Thus, transformation of row and column interactions into factorial contrasts is obtained as:

$$\frac{1}{25}\begin{bmatrix} 4 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -10 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & -2 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 2 & 0 & -4 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 10 & 0 & 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & -10 & 0 \end{bmatrix}\begin{bmatrix} R_L C_L \\ R_L C_Q \\ R_L C_C \\ R_Q C_L \\ R_Q C_Q \\ R_Q C_C \\ R_C C_L \\ R_C C_Q \\ R_C C_C \end{bmatrix} = \begin{bmatrix} AC \\ BD \\ ABCD \\ ABD \\ BC \\ ACD \\ AD \\ ABC \\ BCD \end{bmatrix} .$$

Note that the above is

$$\frac{1}{5}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} R_L \\ R_Q \\ R_C \end{bmatrix} \otimes \frac{1}{5}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} C_L \\ C_Q \\ C_C \end{bmatrix} = \begin{bmatrix} A \\ B \\ AB \end{bmatrix} \otimes \begin{bmatrix} C \\ D \\ CD \end{bmatrix}$$

where $\otimes$ denotes Kronecker product. Note also that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} \text{mean} \\ R_L \\ R_Q \\ R_Q \end{bmatrix} = \begin{bmatrix} \text{mean} \\ A \\ B \\ AB \end{bmatrix}$$

which becomes

$$\frac{1}{5}\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} .$$

The mean row in the above matrix is omitted for the interaction terms only.

If it is desired to express the $R_i C_j$ $(i,j=L,Q,C)$ interactions in terms of the factorial effects, simply multiply the appropriate terms. For example, $R_L C_L = (2A+B)(2C+D) = 4AC + 2AD + 2BC + BD$.

4. Latin Squares of Order n

From the last section, it is easy to see how to approach the problem in general terms for n = a prime power. Suppose that the treatments in two mutually orthogonal latin squares are associated with the levels of two interaction components AB^x and AB^y, for x, y, = 1, 2, ⋯, n-1 and $x \neq y$. Then, the generalized interaction of AB^x and AB^y for any $y \neq 0$ is:

$$AB^x \times AB^y = \sum_{u=1}^{n-1} (AB^x)(AB^y)^u - AB^x - AB^y$$

$$= A + B + AB + \cdots + AB^{n-1} - AB^x - AB^y \quad .$$

The A effect then can be associated with row contrasts, the B effect can be associated with column contrasts, and the remaining n-3 effects can be associated with the treatments in the remaining n-3 pairwise mutually orthogonal latin squares in the set OL(n,n-1). Also, it is immaterial which set of orthogonal contrasts one uses for the n levelsof any effect in the above. For example, one could use orthogonal polynomial coefficients for levels of AB^x and Helmert polynomial for levels of AB^y. The Kronecker product of these individual degree-of-freedom contrasts would produce the $(n-1)^2$ single degree-of-freedom contrasts for the remaining effects in the latin square.

If the single degree-of-freedom row contrasts are denoted as R_0, R_1, ⋯, R_{n-1} and the column single degree-cf-freedom contrasts as C_0, C_1, ⋯, C_{n-1}, then the Kronecker product of these two sets produces the n^2 single degree-of-freedom contrasts in the latin square. R_0C_0 corresponds to the correction for the mean contrast. Likewise, to transform the matrix R for row contrasts into another set of orthogonal contrasts, say F, we need to find the matrix A which does this, i.e., AR = F. Likewise to go from one set, say C, of column contrasts to another set, say G, we need to find a matrix B which transforms C into G, i.e., BC = G. The full set of n^2 contrasts is then obtained as AR $\otimes$ BC = F $\otimes$ G.

For n not a prime power, it is true that any set of treatment contrasts for a latin square or an F-square is still a linear combination of the row-column single degree-of-freedom contrasts, $R_i C_j$, i, j = 1, 2, $\cdots$, n-1. Likewise, for any set of t mutually orthogonal latin squares and/or F-squares, the procedures described previously for going from treatment contrasts to row and column contrasts and vice versa still hold.

5. The F-Square of Order 6

Anderson et al. [1974] found eight mutually orthogonal F-squares with three symbols in each row and column twice. Denote this set of eight as $OF(6;A_1^2,A_2^2,A_3^2;8)$-squares where $n=6$ denotes the square, the three symbols are denoted as A_1, A_2, and A_3, the superscript on the symbol denotes the number of times the symbol occurs in each row and column, and the number $t=8$ denotes the number of mutually orthogonal F-squares. After finding the above, D.A. Anderson and the author independently found another F-square with two symbols which was mutually orthogonal to the above set. The latter obtained this square as the single degree-of-freedom contrast $R_5 C_5$ where $R_5 = C_5 = (1 -1\ 1 -1\ 1\ -1)$ which comes from a particular set of contrasts for the six levels. The nine F-squares will be denoted as $OF(6;A_1^2,A_2^2,A_3^2;8:6;A_1^3,A_2^3;1)$ and are presented in Table 5.1.

The single degree-of-freedom treatment contrasts for these nine mutually orthogonal F-squares are presented in Table 5.2. The interactions among these 17 single degree-of-freedom contrasts should produce the five single degree-of-freedom for row contrasts, the five single degree-of-freedom column contrasts, Column

Row	1	2	3	4	5	6
1	2 0 1 1 0 1 1 0 1	1 1 0 2 0 1 1 0 0	1 2 2 1 1 2 2 1 1	0 0 0 0 1 2 2 1 0	0 2 1 2 2 0 0 2 1	2 1 2 0 2 0 0 2 0
2	2 2 2 2 0 0 2 1 0	0 1 1 0 0 0 2 1 1	1 0 1 2 1 1 0 2 0	2 1 0 1 1 1 0 2 1	0 0 2 1 2 2 1 0 0	1 2 0 0 2 2 1 0 1
3	0 1 1 0 1 2 0 0 1	2 2 2 2 1 2 0 0 0	2 1 0 1 2 0 1 1 1	1 0 1 2 2 0 1 1 0	1 2 0 0 0 1 2 2 1	0 0 2 1 0 1 2 2 0
4	0 2 0 1 1 0 1 2 0	1 0 2 0 1 0 1 2 1	2 2 1 0 2 1 2 0 0	0 1 2 2 2 1 2 0 1	1 1 1 1 0 2 0 1 0	2 0 0 2 0 2 0 1 1
5	1 0 2 0 2 1 0 1 1	0 2 0 1 2 1 0 1 0	0 1 2 2 0 2 1 2 1	2 2 1 0 0 2 1 2 0	2 0 0 2 1 0 2 0 1	1 1 1 1 1 0 2 0 0
6	1 1 0 2 2 2 2 2 0	2 0 1 1 2 2 2 2 1	0 0 0 0 0 0 0 0 0	1 2 2 1 0 0 0 0 1	2 1 2 0 1 1 1 1 0	0 2 1 2 1 1 1 1 1

Table 5.1 OF$(6;A_1^2, A_2^2, A_3^2; 8:6; A_1^3\ A_2^3; 1)$-squares, with the numbering of F-squares being:

1	2	3	4
5	6	7	8
9			

.

Square	Treatment contrast	11	12	13	14	15	16	21	22	23	24	25	26	31	32	33	34	35	36	41	42	43	44	45	46	51	52	53	54	55	56	61	62	63	64	65	66
1	0 vs 1 $=C_1$	0	-	-	+	+	0	0	-	+	0	-	+	+	0	0	-	-	+	+	-	0	+	-	0	-	+	+	0	0	-	-	0	+	-	0	+
	0+1 vs 2 $=C_2$	-2	+	+	+	+	-2	-2	+	+	-2	+	+	+	-2	-2	+	+	+	+	+	-2	+	+	-2	+	+	+	-2	-2	+	+	-2	+	+	-2	+
2	0 vs 1 $=C_3$	+	-	0	+	0	-	0	-	+	-	+	0	-	0	-	+	0	+	0	+	0	-	-	+	+	0	-	0	+	-	-	+	+	0	-	0
	0+1 vs 2 $=C_4$	+	+	-2	+	-2	+	-2	+	+	+	+	-2	+	-2	+	+	-2	+	-2	+	-2	+	+	+	+	-2	+	-2	+	+	+	+	+	-2	+	-2
3	0 vs 1 $=C_5$	-	+	0	+	-	0	0	-	-	+	0	+	-	0	+	-	+	0	+	0	-	0	-	+	0	+	0	-	+	-	+	-	+	0	0	-
	0+1 vs 2 $=C_6$	+	+	-2	+	+	-2	-2	+	+	+	-2	+	+	-2	+	+	+	-2	+	-2	+	-2	+	+	-2	+	-2	+	+	+	+	+	+	-2	-2	+
4	0 vs 1 $=C_7$	-	0	-	+	0	+	0	+	0	-	-	+	+	0	-	0	+	-	-	+	+	0	-	0	+	-	0	+	0	-	0	-	+	-	+	0
	0+1 vs 2 $=C_8$	+	-2	+	+	-2	+	-2	+	-2	+	+	+	+	-2	+	-2	+	+	+	+	+	-2	+	-2	+	+	-2	+	-2	+	-2	+	+	+	+	-2
5	0 vs 1 $=C_9$	+	+	-	-	0	0	+	+	-	-	0	0	-	-	0	0	+	+	-	-	0	0	+	+	0	0	+	+	-	-	0	0	+	+	-	-
	0+1 vs 2 $=C_{10}$	+	+	+	+	-2	-2	+	+	+	+	-2	-2	+	+	-2	-2	+	+	+	+	-2	-2	+	+	-2	-2	+	+	+	+	-2	-2	+	+	+	+
6	0 vs 1 $=C_{11}$	-	-	0	0	+	+	+	+	-	-	0	0	0	0	+	+	-	-	+	+	-	-	0	0	-	-	0	0	+	+	0	0	+	+	-	-
	0+1 vs 2 $=C_{12}$	+	+	-2	-2	+	+	+	+	+	+	-2	-2	-2	-2	+	+	+	+	+	+	+	+	-2	-2	+	+	-2	-2	+	+	-2	-2	+	+	+	+
7	0 vs 1 $=C_{13}$	-	-	0	0	+	+	0	0	+	+	-	-	+	+	-	-	0	0	-	-	0	0	+	+	+	+	-	-	0	0	0	0	+	+	-	-
	0+1 vs 2 $=C_{14}$	+	+	-2	-2	+	+	-2	-2	+	+	+	+	+	+	+	+	-2	-2	+	+	-2	-2	+	+	+	+	+	+	-2	-2	-2	-2	+	+	+	+
8	0 vs 1 $=C_{15}$	+	+	-	-	0	0	-	-	0	0	+	+	+	+	-	-	0	0	0	0	+	+	-	-	-	-	0	0	+	+	0	0	+	+	-	-
	0+1 vs 2 $=C_{16}$	+	+	+	+	-2	-2	+	+	-2	-2	+	+	+	+	+	+	-2	-2	-2	-2	+	+	+	+	+	+	-2	-2	+	+	-2	-2	+	+	+	+
9	0 vs 1 $=C_{17}$	-	+	-	+	-	+	+	-	+	-	+	-	-	+	-	+	-	+	+	-	+	-	+	-	-	+	-	+	-	+	+	-	+	-	+	-

Table 5.2. Seventeen single degree of freedom contrasts for treatments from the F-squares in Table 6.1.

and a <u>remaining</u> set of seven degrees of freedom which can be used to construct additional orthogonal F-squares. It should be possible to produce a complete set of mutually orthogonal F-squares for <u>any</u> n. Note that these remaining seven interaction degrees of freedom are orthogonal to rows and columns in the n by n square. A check should be made to ascertain that the interaction of the treatments in two $F(6;A_1^2,A_2^2,A_3^2)$-squares is orthogonal to the remaining 17 - 4 = 13 single degree of freedom contrasts for treatments from the remaining seven F-squares.

On checking the interactions of the eight sets of two and one set of one to make 17 single degree of freedom contrasts in Table 5.2 to ascertain which are orthogonal to row and column contrasts, we note that from the interaction sum of squares from treatments in square 1 with the treatments in square 9, we obtain contrasts with four different coefficients and from these write the following square from contrasts $C_1 \times C_{17}$ and $C_2 \times C_{17}$:

0	1	2	2	1	3
3	1	1	0	2	2
1	3	0	1	2	2
2	2	3	1	1	0
2	2	1	3	0	1
1	0	2	2	3	1

This $F(6;A_1^2,A_2^2,A_3^2,A_4)$-square is orthogonal to squares 5, 6, 7, and 8, it is composed from square 1 plus square 9, and it is not orthogonal to squares 2, 3, and 4. The following $F(6;A_1\ A_2^2\ A_3^2\ A_4)$-square is orthogonal to the same squares as above (changes are in parentheses):

0	1	2	1(2)	2(1)	3
3	2(1)	1	0	1(2)	2
2(1)	3	0	1	2	1(2)
1(2)	2	3	2(1)	1	0
2	1(2)	2(1)	3	0	1
1	0	1(2)	2	3	2(1)

From the interactions $C_{10} \times C_{17}$ and $C_9 \times C_{17}$ we obtain the following $F(6;A_1^2,A_2^2,A_3^2,A_4^2)$-square:

1	2	2	1	0	3
2	1	1	2	3	0
2	1	0	3	1	2
1	2	3	0	2	1
0	3	1	2	2	1
3	0	2	1	1	2

This square which is composed from squares 5 and 9, is orthogonal to squares 1 to 4 and 6 to 8. As in the previous case the above $F(6;A_1,A_2^2,A_3^2,A_4)$-square could be replaced by the following and the orthogonality with the other squares would still hold (changes are in parentheses):

2(1)	1(2)	2	1	0	3
1(2)	2(1)	1	2	3	0
2	1	0	3	2(1)	1(2)
1	2	3	0	1(2)	2(1)
0	3	2(1)	1(2)	2	1
3	0	1(2)	2(1)	1	2

The fact that squares 5 and 9 can be put together to form an $F(6;A_1,A_2^2,A_3^2,A_4)$-square suggests that one might superimpose square 9 on square 5 to form this square as follows:

01 - 1	00 - 2	11 - 3	10 - 4	21 - 5	20 - 6
00 - 2	01 - 1	10 - 4	11 - 3	20 - 6	21 - 5
11 - 3	10 - 4	21 - 5	20 - 6	01 - 1	00 - 2
10 - 4	11 - 3	20 - 6	21 - 5	00 - 2	01 - 1
21 - 5	20 - 6	01 - 1	00 - 2	11 - 3	10 - 4
20 - 6	21 - 5	00 - 2	01 - 1	10 - 4	11 - 3

We note that there are six different combinations of a 2 x 3 factorial which suggest a latin square of order 6. This latin square is orthogonal to $F(6;A_1^2,A_2^2,A_3^2)$-squares numbers 1 to 4 and 6 to 8. This means that a set $OF(6;A_1^2,A_2^2,A_3^2,7: 6;A_1,A_2,A_3,A_4,A_5,A_6;1)$-squares has now been formed and the treatment degrees of freedom in these 8 orthogonal F-squares account for $7(3-1) + 1(6-1) = 19$ of the 25 row by column interaction degrees of freedom. If one can account for the remaining six degrees of freedom, a CSOFS of order six exists.

The preceding exercise leads to a conjecture:

Conjecture: If the interaction contrasts between the treatment contrasts in two (or more) F-squares are unconfounded with row and column effects, this means that an F-square with more symbols can be formed from these two (or more) F-squares.

This conjecture is related in some unknown manner to Mandeli's [1975] concept of "closure under multiplication". From the preceding we also can make the following statement:

Statement: Any latin square of order six can be decomposed into an $F(6;A_1^3,A_2^3)$-square and an $F(6;A_1,A_2,A_3,A_4,A_5^2)$-square as well as into an $F(6;A_1^3,A_2^3)$-square, an $F(6;A_1^2,A_2^2,A_3^2)$-square, and an unknown remainder.

The proof of this statement comes directly from rewriting the numbers 1, 2, 3, 4, 5, 6 as 00, 01, 02, 10, 11, 12 in any latin square. Then in the ij'th, i=0,1, j=0,1,2, combination notation, the i'th subscript forms the $F(6;A_1^3,A_2^3)$-square, the j'th subscript forms the $F(6;A_1^2,A_2^2,A_3^2)$-square, and the $F(6;A_1,A_2,A_3,A_4,A_5^2)$-square is formed from the remaining orthogonal contrasts as follows:

Treatment 1	2	3	4	5	6	
-	+	-	+	-	+	forms an $F(6;A_1^3,A_2^3)$-square
+	+	+	+	-2	-2	$\}$ forms an $F(6;A_1^2,A_2^2,A_3^2)$-square (rows 2–3)
+	+	-	-	0	0	
+	-	+	-	0	0	forms an $F(6;A_1,A_2,A_3,A_4,A_5^2)$-square (rows 2–5)
+	-	-	+	0	0	

The statement can obviously be generalized for any latin square of order $n = qkrs \cdots$.

Having exhausted the "interaction of treatment contrasts which are orthogonal to rows and columns" approach, one could consider linear combinations of treatment interaction contrasts which are orthogonal to row and column effects. For example, contrasts

$$(C_1 \times C_4 - C_2 \times C_3)/2 \quad \text{and} \quad (3(C_1 \times C_3) + C_2 \times C_4)/2$$

can be used to form an $F(6;A_1^2,A_2^2,A_3^2)$-square as follows:

0	2	0	2	1	1
2	0	1	1	2	0
0	2	1	1	0	2
1	1	2	0	2	0
1	1	0	2	0	2
2	0	2	0	1	1

This F-square is orthogonal to squares 1 to 4 but not to 5 to 9, and is called square 10.

Likewise,

$$(C_1 \times C_6 - C_2 \times C_5)/2 \quad \text{and} \quad (3(C_1 \times C_5) + C_2 \times C_6)/2$$

can be used to form the following $F_{11}(6;A_1^2,A_2^2,A_3^2)$-square:

1	1	0	2	0	2
2	0	2	0	1	1
0	2	0	2	1	1
2	0	1	1	2	0
0	2	1	1	0	2
1	1	2	0	2	0

Square 11 is orthogonal to squares 1 to 4 but not to squares 5 to 10. Also, $(C_1 \times C_8 - C_2 \times C_7)/2$ and $(3(C_1 \times C_7) + C_2 \times C_8)/2$ can be used to form an $F(6;A_1^2,A_2^2,A_3^2)$-square and so forth. Because of the weakness of linear model theory here, it is not known how to proceed with linear combinations of interaction contrasts for usefulness in constructing orthogonal F-squares.

The complete set of treatment contrasts which are orthogonal to each other, to rows contrasts and to columns contrasts and are not in the same F-square listed in Table 5.1, are:

C_1C_{17} - which can be used to form an $F(A_1^2,A_2^2,A_3^2)$-square
C_2C_{17}
C_9C_{17} - which can be used to form an $F(A_1^2,A_2^2,A_3^2)$-square
C_9C_{18} - which can be used to form an $F(A_1^2,A_2^2,A_3^2)$-square
$C_9C_{10}C_{17}$ - which can be used to form an $F(A_1^2,A_2^2,A_3^2)$-square
$C_9C_{17}C_{18}$ - which can be used to form an $F(A_1^2,A_2^2,A_3^2)$-square
$C_{10}C_{17}$
$C_{10}C_{17}C_{18}$
$C_{17}C_{18}$

where C_{18} is obtained from the following $F(A_1^4,A_2^2)$-square:

1	1	1	1	0	0
1	1	1	1	0	0
1	1	0	0	1	1
1	1	0	0	1	1
0	0	1	1	1	1
0	0	1	1	1	1

The remaining interaction contrasts are confounded with row effects, with column effects, or with both row and column effects. It is not known how to use contrasts C_2C_{17}, $C_{10}C_{17}C_{18}$, and $C_{17}C_{18}$ to form F-squares.

6. Interactions of Row and Treatment and Column and Treatment Contrasts

Since the results in section 5 appear to indicate only that larger F-squares rather than additional F-squares can be obtained, we shall study row by treatment and column by treatment contrasts. Only those which are unconfounded with columns and rows will be considered. The row and column contrasts used were:

Row	1	2	3	4	5	6
R_1	1	-1	1	-1	1	-1
R_2	2	-2	-1	1	-1	1
R_3	-2	-2	1	1	1	1
R_4	0	0	-1	1	1	-1
R_5	0	0	-1	-1	1	1

Column	1	2	3	4	5	6
K_1	1	-1	1	-1	1	-1
K_2	2	-2	-1	1	-1	1
K_3	-2	-2	1	1	1	1
K_4	0	0	-1	1	1	-1
K_5	0	0	-1	-1	1	1

The treatment contrasts were those in Table 5.2. One additional treatment contrast was added, i.e. C_{18}, and was obtained from the following $F(6;A_1^4A_2^2)$-square:

1	1	1	1	0	0
1	1	1	1	0	0
1	1	0	0	1	1
1	1	0	0	1	1
0	0	1	1	1	1
0	0	1	1	1	1

The following contrasts were orthogonal to row and column contrasts:

R_1C_1 which produces an $F(6;A_1^2,A_2^2\ A_3^2)$-square

R_1C_2

R_1C_9

R_1C_{10}

R_1C_{18}

$\left.\begin{array}{r} R_1C_9C_{17} \\ R_1C_{10}C_{17} \end{array}\right\}$ which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

R_2C_9

R_2C_{10}

R_2C_{17}

R_2C_{18}

R_3C_{17}

$R_3C_9C_{17}$

$R_3C_{10}C_{17}$

R_4C_9

R_4C_{10}

R_4C_{17} which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

R_4C_{18}

R_5C_{17} which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

$R_5C_9C_{17}$

$R_5C_{10}C_{17}$

It is not known what type of F-square can be produced from those not marked above.

The interaction of column contrasts with treatment contrasts $C_1, C_2, \cdots, C_{18}$ which are orthogonal to rows and columns are:

K_1C_1 which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

K_1C_2

$\left.\begin{matrix} K_1C_9 \\ K_1C_{10} \end{matrix}\right\}$ which produce an $F(6;A_1^2,A_2^2,A_3^2)$-square

K_1C_{11} which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

K_1C_{12}

K_1C_{13} which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

K_1C_{14}

K_1C_{15} which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

K_1C_{16}

K_1C_{18}

$K_1C_9C_{17}$ which produces an $F(6;A_1^2,A_2^2,A_3^2)$-square

$K_1C_{10}C_{17}$

K_2C_9

K_2C_{10}

K_2C_{11}

K_2C_{12}

K_2C_{13}

K_2C_{14}

K_2C_{15}

K_2C_{16}

K_2C_{17}

K_2C_{18}

K_3C_{17}

$K_3C_9C_{17}$

$K_3C_{10}C_{17}$

K_4C_9

K_4C_{10}

K_4C_{11}

K_4C_{12}

K_4C_{13}

K_4C_{14}

K_4C_{15}
K_4C_{16}
K_4C_{17}
K_4C_{18}
K_5C_{17}
$K_5C_9C_{17}$
$K_5C_{10}C_{17}$

It is not known how to use the unmarked contrasts above to form F-squares.

There are row by treatment contrasts which are also column contrasts. The complement of those contrasts orthogonal to columns are confounded with columns. Such contrasts as R_1C_3, R_1C_4, R_1C_5, R_1C_6, R_1C_7, R_1C_8, R_1C_{11}, R_1C_{12}, etc. are confounded with column effects. The same is true for the column by treatment contrasts. We have listed the contrasts above which are orthogonal to row contrasts, and the complement of these, or the remaining ones, are confounded with row contrasts.

7. Some Unsolved Problems

From the preceding it appears that the following conjecture is true:

Conjecture: Let C_{ij}, $i=1,2,\cdots,n$, $j=1,2,\cdots,n-1$ be the coefficients of n-1 mutually orthogonal contrasts among n items. Then, for any j, say j^*, the set of k, $1 \le k \le n-1$, distinct contrasts in the set C_{ij^*}, $C_{ij}\ C_{ij^*}$, $j \ne j^*$, are mutually orthogonal. Also, for any two j, say j^* and j^+, the set of k, $1 \le k \le n-1$, distinct contrasts in the set C_{ij^*}, C_{ij^+}, $C_{ij^*}C_{ij^+}C_{ij}$, $j \ne j^+ \ne j^*$, are mutually orthogonal. This can be extended for any set of j .

The start of a proof can be made by noting that $\sum_{i=1}^{n} C_{ij}C_{ij^*} = 0$, $j \ne j^*$, from the orthogonality of the C_{ij} and hence is a contrast. From the Helmert polynomial contrasts we observe that there are one to n-1 distinct contrasts formed, depending upon which j^* is selected. But, how does one prove that $\sum_{i=1}^{n} C_{ij}^2 * C_{ij} = 0$ or $\sum_{i=1}^{n} (C_{ij}C_{ij}*)(C_{ij'}C_{ij}*) = 0$ for $j \neq j^* \neq j'$?

The weakness of linear models theory in this area is again exemplified by the following questions:

1. How should one select row (R_i), column (C_j), and treatment contrasts in order to obtain the simplest relation between R_iC_j and the treatment contrasts in a set of orthogonal F-squares?

2. The interaction of treatment contrasts in different F-squares should produce what kinds of contrasts?

3. If one interacts treatment contrasts in two orthogonal F-squares and if these contrasts are orthogonal to rows, to columns, or to both, what does this mean?

4. What conclusions can be drawn from interaction contrasts among treatment contrasts from a set of t F-squares?

5. How does one use Mandeli's [1975] concept of "closure under multiplication" for $n \neq$ a prime power?

6. How does one generalize the works of Finney [1945, 1946a, 1946b, 1960], Freeman [1964, 1966], and Gilliland [1975] on orthogonal partitions of latin squares? For example, theorem 3 in Finney [1945] can be easily expanded and generalized (see section 5).

7. How does one tie up all this work with the works of A. Hedayat and E. Seiden on F-squares?

Obviously these questions could go on and on but these are sufficient to indicate that much work needs to be done in this area in order to form a complete and coherent theory.

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